27. (Cancelled)

Remarks

[1] Claims 1-27 are pending in this application. Claims 1-27 stand rejected. Claims 2-4, 21-23 and 27 are cancelled. As a courtesy to the Examiner, the Applicants have used the paragraph structure used by the Examiner to respond to the Examiners requests, objections and rejection. Claims 12-19 are amended to place them in condition for allowance based on the Examiner's statement that they are non-obvious over the prior art. Certain other claims are amended to place them in better form in case of the necessity of an appeal. No new limitations are included in the amendments; so if Examiner does believe they are currently allowable despite Applicants' firm belief that they are, it is respectfully requested that they be entered for purposes of narrowing the issues for an appeal.

Priority

[2] The Examiner is correct that the Supplemental Application Data Sheet is in error. A corrected Supplemental Application Data Sheet referencing the provisional application is submitted with this amendment as Appendix I.

Drawings

[3] The Examiner has objected to the drawings on several grounds. To overcome each the Applicants have done the following:

[3-1] FIG. 6 is amended to substitute "Fail Rate" for "Fall Rate" per the originally submitted drawings. In FIG. 8 "CPU" is substituted for "RAM" for reference character 10 as per the originally submitted drawings.

[3-2] In FIG. 4 the drawing is corrected to remove the first reference character 58, substituting "56" per the originally submitted drawing.

The Applicants apologize that the formal drawing contained these mistakes and believe the Examiner's rejection is overcome.

Specification

[4] The Applicants thank the Examiner for withdrawing the objections to the specification.

Claim Rejection – 35 USC §112, First Paragraph

[5] The Examiner has rejected the application under 35 USC 112, first paragraph, arguing that there is a substantial lack of teaching for the claimed invention.

[6] In the rejection states the Examiner states "how to 'combine the individual values and then add hot-e input based on performance sort" have not been described in the specification."

The Applicants disagree. As stated in the previous amendment the Monte Carlo analysis, which Applicants are using to 'combine the individual values and then add hot-e input based on performance sort' is well known in the art as demonstrated by readily available

textbook sources. To prove this point Applicants have attached in Appendix II three common text book references discussing Monte Carlo analysis in engineering applications. For the first two references only the title pages are provided. The third source, Probabilistic Systems Analysis, presents a practical discussion and application of the Monte Carlo technique. (Note that the author subtitles it as an "Introduction to Probabilistic Models.") Applicants have attached several pages from that text. The Examiner should particularly note the flow chart on page 157. It describes the basic steps behind the Monte Carlo analysis, and forms the basis for Applicant's FIG. 7. Please note also that Figure 7-6 on p. 173, which shows the output of a particular Monte Carlo analysis. (See FIG. 2.) Simply put Applicants are applying a known statistical technique to solve a guardband problem in a novel way by accounting for the tester environment, tester-to-system offset, system environment, and reliability parameters (e.g. hot-e degradation).

In addition, Figure 7 and the text referencing Figure 7 describes Applicants' preferred Monte Carlo analysis of selecting a value from each distribution type (tester environment, tester-to-system offset, and system environment) and then combining (i.e. adding) those values with a hot-e value (based on performance sort) to determine a guardband value for each iteration of the analysis. Applicants' Monte Carlo function, selecting a value from each distribution type and combining those values with a hot-e value, is comparable to the generic Monte Carlo function as illustrated on p. 157 of the third source, Probabilistic Systems Analysis, where the Monte Carlo function is depicted as "Compute $y_i = g(x_1 \ldots x_{20})$." This process is continued for n iterations in accordance with the Monte Carlo approach at which time the output is the distribution of guardbands as illustrated in Figure 2.

Each value contributed from the tester environment distribution model contributes a performance value (e.g. FMAX) as described on page 3, lines 12 on. Thus, when the hot-e value is added to the combined distribution values for a particular iteration of the Monte Carlo analysis, the hot-e value added to these combined values corresponds to the

performance value (FMAX) as taught at page 6, lines 8-19 and in accordance with Figure 6.

In conclusion, the Applicants specification is in full compliance with 35 U.S.C., first paragraph.

Claim Rejections – 35 U.S.C. 112, 2nd Paragraph

[7] The Examiner rejected claims 2-3, 12-13 and 15-17 under 35 USC 112, second paragraph.

[8] Applicants have cancelled claims 2 and 3 and amended the other claims to overcome the Examiner's rejections. Claim 1 now includes claim 2 and has an antecedent basis added for "product." Claim 16 now is dependent on claim 13 which contains the referenced step. Claim 20 is amended so that "product" has an antecedent basis in the preamble.

Claim Rejections – 35 U.S.C. 102

[10] The Examiner rejected now pending claims 1, 6-7, 20-24 and 27 under 35 USC 102(b) as being anticipated by Mittle et al., U.S. Patent 5, 634,001 issued May 27,1997 ("Mittle").

[10-2, 3, 4] Mittle does not teach or suggest modeling the tester environment or end-system environment in which the microprocessor will reside. Claim 1 has been amended to introduce in the claim **both the system on which the product is used and tester offset as variables**. Mittle teaches a method of determining a hot-e guardband by taking into consideration circuit timing data, operating voltage and case temperature as well as the voltage / frequency response of a microprocessor.
These are all internal characteristic of the of the "product," ie, microprocessor or inputs to the microprocessor. They are not the "system" in which the product is used as used and defined by applicants. Nor does Mittle speak of tester to system offset. Nowhere in the section sited by the Examiner is there any mention of system to tester offset especially in terms of what applicants clearly refers to and claims as its invention. Mittle restricts its teaching to the calculation of a hot-e guardband which is only one consideration of the present invention.

As stated previously and repeated here, Mittle does not teach or suggest modeling of "system variables." Instead, Mittle only considers a nominal operating voltage and temperature at which a microprocessor is expected to operate when determining a hot-e guardband. Therefore Mittle does not teach "creating a set of **distribution** models representative of variables" in claim 1.

As stated previously and repeated here, Mittle simply determines the difference expected in a microprocessor's performance between beginning of life (BOL) and end of life (EOL). Mittle does not teach or suggest that the test environment upon which a microprocessor is tested should be modeled to account for variations within that system. Hence there is no teaching of a variable for "system to tester offset" as set forth in claim 1. Mittle discloses circuit-level timing analysis. This is based upon a simulation of a particular circuit. Mittle does not teach or suggest the use of a tester to determine actual circuit performance. A simulation environment on a computer does not inject variations into guardband determinations as does an actual tester environment. An actual tester environment introduces many variations as a result of both electrical and mechanical components. These components do not exist when simulating circuit performance, and thus there is no need to model variations that may exist in a tester environment. So there is no teaching of using the "test system" as a variable.

[10-5] The arguments with regards to claim1 render claim 6 allowable.

[10-6] As stated previously and as preserved here, Mittle does not teach or suggest a sample of at least 10. In fact, there is no sample size requirement in Mittle. The part of Mittle cited by the Examiner discusses a histogram of propagation delays for all the paths of a microprocessor. There is no **sampling** as set forth in claim 7, just simply a graphical representation of the delay associated with each path of a microprocessor.

[11-7] As stated previously and as persevered here, Mittle does use a reliability wearout model it does not teach or suggest the use in the context of this invention; i.e., of a "set of distribution models representative of variables that affect the specification." So claim 10 is allowable in view of how it the model is used.

[10-8] As stated previously and as preservered here, the Examiner notes that claim 1 has similar limitations to those in claims 20. Therefore the arguments overcoming anticipation for that claim applies equally well to claim 20.

[10-9] Claim 27 is cancelled.

[11] The Examiner also claims 1, 8-99, 20-23 and 27 under 35 USC 102(b) as being anticipated by Conrad, et al., Calculating Error of Measurement on High Speed Microprocessor Test," Proceeding of International Test Conference, October 1994, pages 793-801 ("Conrad").

[11-1] Claim 1 now contains the limitations in claims 2 and 3. Thus claim 1 is allowable over Conrad.

[11-2,3] Claims 8-9 now rely on claim 1 which in its new form is allowable over Conrad.

In lieu of the above arguments, it is clear that neither Mittle or Conrad teach what is claimed by Applicants.

Claim Rejections – 35 U.S.C. 103

[13] Claims 5 and 25 are rejected under 35 USC 103(a) in view of Mittle and Applicants assertion.

[13-1] As stated previously and as preserved here even if it would be obvious to use a Monte Carlo analysis in light of Mittle, this is irrelevant because Mittle teaching is limited to how to determine a guardband for reliability wearout mechanisms. Mittle fails to suggest the use of models (especially models relating to use of the product in a system and tester offset that are representative of variables in a specification. See discussions regarding claims 1 and 20 above. As to the Monte Carlo analysis, Mittle actually teaches away from the use of a Monte Carlo analysis because such an analysis would inject unnecessary uncertainty into the guardband calculation. The techniques taught in Mittle have no use for a statistical sampling technique like that taught and claimed by Applicants. Statistical sampling may actually cause excessive variation in the hot-e guardband calculation, thus rendering it ineffective. Monte Carlo analysis is suited for applications where a probabilistic approximation to a solution is desired. Mittle teaches how to determine an exact guardband based upon circuit timing analysis, operating voltage and temperature and channel length. A probabilistic approximation is not desirable in this situation because nothing needs to be approximated.

As stated previously and preserved here, Applicants assertions regarding the variability of the type of statistical analysis tools in its own claimed invention has nothing to do as to whether it would be obvious to include such tools in Mittle.

Since the Examiner has failed to establish that Mittle or Conrad reads on claims 5 and 25, let alone the claims upon which they depend, the Examiners 103 arguments with respect to those claims fail.

[14] The Examiner has rejected claims 11 and 26 under 35 USC 103 as being unpatentable over Mittle ii\n view of Kreyszig, "Advanced Engineering Mathematics," John Wiley & Sons, 1988, pages 1248-1253 ("Kreyszig").

[14-1] With regard to the remaining 103 rejections, the combinations cited by the examiner fail to suggest three of the novel aspects of the present invention set forth in claims 11 and 26. First, there is no suggestion of the use of "distribution models representative of variables that affect a **specification**." Furthermore, there is no suggestion that the distribution models should be analyzed using a statistical tool. The cited art only deals with individual components of a guardband (reliability wearout mechanisms in Mittle). There is no suggestion as to how one would determine an overall guardband based upon reliability wearout mechanisms as well as variations **within both the tester and system in which the product is used.** Finally, there is no suggestion to select a final guardband based upon a statistical analysis.

For these reasons claims 11 and 26 are clearly patentable over the prior art.

[15] Applicants thank the Examiner for finding claims 12-19 non-obvious over the prior art. These claims now stand alone and are in condition for allowance.

SUMMARY AND CONCLUSION

In view of the foregoing, withdrawal of the rejections and the allowance of the current pending claims is respectfully requested. If the Examiner feels that the pending claims could be allowed with minor changes, the Examiner is invited to telephone the undersigned to discuss an Examiner's Amendment.

Respectfully submitted,

Date: Oct 4, 2004

BY: Richard M. Kotulak
Attorney for the Applicants
Registration No.: 27,712
Telephone: 802-769-4457

Practitioner's Docket No. BUR990230US1 **PATENT**

IN THE UNITED STATES PATENT AND TRADEMARK OFFICE

In re prior application of: Mark R. Bilak

Application No.: 09 / 520,257 Group No.: 2123
Filed: 03/07/2000 Examiner: Day, Herng-Der
For: STATISTICAL GUARDBOND METHODOLOGY

Mail Stop Patent Application
Commissioner for Patents
P.O. Box 1450
Alexandria VA 22313-1450

SUPPLEMENTAL APPLICATION DATA SHEET
37 C.F.R. § 1.76(c)

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(2) Should identify the information that is being changed (added, deleted, or modified) and therefore need not contain all the previously submitted information that has not changed." 37 C.F.R. § 1.76(c).

The following information on the Application Data Sheet is changed as indicated:

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BIBLIOGRAPHIC DATA

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Title of Invention:

Docket number assigned to this application:

Suggested Classification: Class:

Subclass:

Technology Center to which subject matter is assigned:

NOTE: "The suggested classification and Technology Center information should be supplied for provisional applications whether or not claims are present. If claims are not present in a provisional application, the suggested classification and Technology Center should be based upon the disclosure." 37 C.F.R. § 1.76(b)(3).

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Type of application:

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  - ☐ application is to be published
    Suggested drawing figure for publication: ____________________
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- ☐ plant
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    of plant being claimed.
- ☐ design
- ☐ reissue
- ☐ provisional

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- ☐ discloses a significant part of the

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☒ Domestic priority for this application is claimed as follows:

☒ 35 U.S.C. § 119(e): Application No.: 60/172,198
Filed: December 17, 1999
Status: Abandoned
Relationship: Provisional for Non-Provisional

☐ 35 U.S.C. § 120: Application No.: ______
Filed: ______
Status: ______
Relationship: ______

☐ 35 U.S.C. § 121: Application No.: ______
Filed: ______
Status: ______
Relationship: ______

☐ 35 U.S.C. § 365(c): Application No.: ______
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☐ Foreign priority is claimed for this application as follows:

Country: ________________

Application No.: ________________

Filing date: ________________

Status: ________________

Foreign application having a filing date before that of the above application for which priority is claimed.

☐ None

☐ Country: ________________

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The assignee(s) of this application is/are:

Name of assignee: ________________

Address of assignee: ________________

Extent of interest of assignee in application: ________________

Reg. No. 27712

Signature of Practitioner

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1

PARAMETER ESTIMATION IN ENGINEERING AND SCIENCE

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McGRAW-HILL BOOK COMPANY, INC. 1961
NEW YORK TORONTO LONDON

 3A

PROBABILISTIC SYSTEMS ANALYSIS

AN INTRODUCTION TO PROBABILISTIC MODELS, DECISIONS, AND APPLICATIONS OF RANDOM PROCESSES

ARTHUR M. BREIPOHL

PROBABILISTIC SYSTEMS ANALYSIS

AN INTRODUCTION TO PROBABILISTIC MODELS, DECISIONS, AND APPLICATIONS OF RANDOM PROCESSES

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30

Thus independence is not required for the first result. Now

$$E[(Y-\mu_Y)^2] \simeq E\left\{\left[\sum_{i=1}^{n} a_i(X_i-\mu_i)\right]^2\right\}$$

$$\simeq \sum_{j=1}^{n}\sum_{i=1}^{n} a_i a_j E[(X_i-\mu_i)(X_j-\mu_j)].$$

$$\sigma_Y^2 \simeq \sum_{i=1}^{n} a_i^2 \sigma_{X_i}^2 + \sum_{\substack{j=1 \\ i \neq j}}^{n}\sum_{i=1}^{n} a_i a_j \sigma_{ij} \qquad (6\text{-}18)$$

where $\sigma_{ij} = E[(X_i-\mu_i)(X_j-\mu_j)] = E[X_iX_j] - \mu_i\mu_j.$

6-8 SYNTHETIC SAMPLING (MONTE CARLO TECHNIQUE)

The methods discussed so far to find the distribution of

$$Y = g(X_1, X_2, \ldots, X_n)$$

have been approximations or too involved to be practical for large problems. In this section a very simple and intuitively satisfying method is presented. Its only drawbacks are that it requires a digital computer and general parametric results are not obtained, thus limiting the applicability to synthesis.

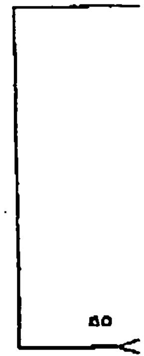
It is assumed that $Y = g(X_1, \ldots, X_n)$ is known and that the joint density $f_{X_1,X_2,\ldots,X_n}$ is known. Now if a sample value of each random variable were known (say $X_1 = x_{11}, X_2 = x_{12}, \ldots, X_n = x_{1n}$), then a sample value of Y could be computed [say $y_1 = g(x_{11}, x_{12}, \ldots, x_{1n})$]. Then if another set of sample values were chosen for the random variables (say $X_1 = x_{21}, \ldots, X_n = x_{2n}$), then $y_2 = g(x_{21}, x_{22}, \ldots, x_{2n})$ could be computed.

If one had the time one could compute many such sample values of Y. The computer actually supplies the speed that makes many such calculations possible. There is just one problem. How does the computer select the different values of $X_1, X_2, \ldots, X_n$?

If each of the random variables had a uniform distribution between 0 and 1, numbers for each random variable could be chosen from a table of random numbers. Actually, computer routines generate pseudorandom numbers which may be used.

Consider the following case. Let the random variables $X_1, X_2, \ldots, X_{20}$ be independent and each uniformly distributed between zero and one, and let $Y = g(X_1, X_2, \ldots, X_{20})$ be a known function. Then the computer program

to compute an
ing basic steps



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TECHNIQUE)

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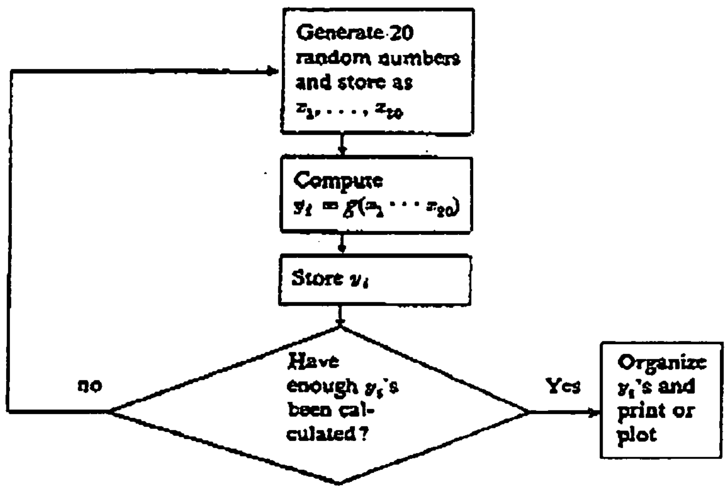
$X_1, X_2, \ldots, X_{20}$ ero and one, and :omputer program

to compute an approximation to the distribution of Y consists of the following basic steps.



If a plot is desired it may be convenient to plot a usual bar chart or histogram. This method simply calls for breaking the range of Y into say 30 mutually exclusive cells of the same size and plotting vertically the number of samples that fell into that cell. (See Chapter VIII for more details of histograms.)

The case of uniformly distributed variables was considered. Now let X_i have a distribution function F_{X_i}. To obtain a random sample of X_i the following procedure may be used. Select a random sample of U which is uniformly distributed between 0 and 1. Call this random sample u_1. Then $F_X^{-1}(u_1)$is the random sample of X (see Example 6-7).

For example, suppose that X is uniformly distributed between 10 and 20. Then

$$\begin{aligned} F_{X_i}(x) &= 0, && x < 10, \\ &= (x - 10)/10, && 10 \leq x < 20, \\ &= 1, && x \geq 20. \end{aligned}$$

This is shown in Figure 6-29.

Notice $F_X^{-1}(u) = 10u + 10$. Thus if the value .250 were the random sample of U, then the corresponding random sample of X would be 12.5.

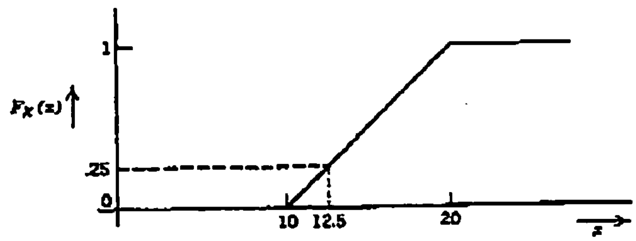


Figure 6-29

As another example suppose that X is normally distributed with mean 0 and variance 1. Then a random value of .250 for U would correspond to $-.67$ for a value of X. This result follows from a table of the normal distribution function. Practically, most computers automatically generate standard normal random variables. If this is the case then random sample of a normal random variable Y with mean μ and a variance σ^2 may be generated from a standard normal random variable X by recalling that

$$X = \frac{Y - \mu}{\sigma}$$

or

$$Y = X\sigma + \mu. \tag{6-19}$$

Equation 6-19 can be easily checked by finding the characteristic function of Y.

The only difference when the random variables are dependent is that the dependence must be taken into account when the random samples are generated. We assume that the dependence is expressed in terms of conditional distributions. If this is not the case the joint distributions can always be reduced to the required conditional distributions.

For illustration consider three dependent random variables: X_1, X_2, and X_3. We first generate a random sample of X_1 by the same methods discussed above. Call this sample x_{11}. We next generate a random sample of X_2 using $F_{X_2|X_1=x_{11}}$, by the same method used before. Call this sample x_{12}. Then we use $F_{X_3|X_1=x_{11},X_2=x_{12}}$ to generate x_{13}. Thus nothing changes except that the conditional distribution functions are used in generating the random samples.

With a fast digital computer thousands of simulations can be run in reasonable times. Monte Carlo solutions often involve 10,000 or more simulations. An example is given in the next chapter.

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6-9 SUMMARY

The purpose of this chapter was to consider the important engineering problem of finding the distribution of $Y = g(X_1, X_2, \ldots, X_n)$ where the distribution of the X_i's is known.

First the problem of $Y = g(X)$ was considered and this problem was solved by (6-2). Examples were given to illustrate its application.

Next $Y = \sum_{i=1}^{n} X_i$ was considered, and it was shown that in the case of independent random variables the solution involved convolution of the density functions or multiplication of the characteristic functions. The case where Y is a linear combination of the X_i's was shown to be only a slight generalization of this problem.

The general problem was then considered and although a general method of solution was outlined, the difficulty of solution was illustrated by an example, and approximations were suggested.

Two approximations for the general solution were described. First a Taylor series approximation, moments, and the central limit theorem were used. Then a Monte Carlo method was suggested.

References B1, D3, D4, and P2 provide additional reading.

6-10 PROBLEMS

1. X has a normal density function with mean 1 and variance 2.

$$Y = \tfrac{1}{2}X - 1.$$

Find the density of Y.

2. The power P dissipated in a resistor is $P = I^2R$. Assume $R = 2$ and I is a random variable with a normal density.

$$f_I(i) = \frac{1}{\sqrt{2\pi}} e^{\frac{-i^2}{2}}.$$

Find the density function of P.

3. The output of a full wave rectifier is $Y = |X|$. Find the density function of Y when X has a uniform density from -1 to $+1$.
4. The output of a square law detector is

$$Y = aX^2, \quad a > 0.$$

Find the density function of Y in terms of the density function f_X of X.

is usually a good assumption. When the standard deviation of the parts are known, then the mean and variance of the output can be computed using the approximation developed in the last chapter.

Again something must be assumed to describe the output distribution and the probability of being out of tolerance. It is suggested that if enough variables are involved and the function is approximately a linear combination, then a normal density may be assumed. Then the probability of being outside tolerance limits can be computed.

We now show an example of a tolerance problem which is solved using a Monte Carlo approach.

Example 7-2

This example, a simplification of a problem that actually occurred in practice, was worked by the method shown, and the results actually obtained in manufacture corresponded with the theoretical results.

The simplified version is shown in Figure 7-4. The bar will fit within the bracket if $Y_a < Y_b$ and there will be interference (or no fit) if $Y_b < Y_a$. In the actual case Y_a and Y_b involved 41 dimensions and the configuration was more complicated than simply a sum of lengths. Actually the equations for Y_a and Y_b involved arcs and angles, thus various trigonometric functions were involved.

The problem was solved by finding the probability distribution of $Z = Y_b - Y_a$ by Monte Carlo sampling. Note that if $Z > 0$ there is no problem, while if $Z < 0$ there will be no fit and the parts cannot be assembled.

The problem arose because just as production and assembly were about to start it was discovered that interference was possible. (As customary, the intent was to tighten the tolerances on each part until no interference is possible at the worst case, but the designer made a mistake in his worst case calculations.) Then the question was, what is the probability of interference? If it is low enough then it would be better to have a few that would not fit, rather than wait and spend the extra money to redesign some of the parts.

To find the probability density of Z via a Monte Carlo technique, one must have $Z = g(X_1, \ldots, X_{41})$ and must know the joint distribution of

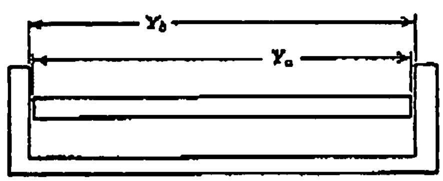

Figure 7-4

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$X_1, \ldots, X_{21}$. The function g was found from the drawings using trigonometry. A part of the equation is shown in Figure 7-5 simply to illustrate the

$$
\begin{aligned}
Y = (R + W)\sin\Bigg\{ & \theta - \Bigg[\sin^{-1}\left(\frac{H_{21}}{R}\right) + 2\sin^{-1} \\
& \times \left(\frac{\sqrt{\left(\sqrt{r^2 - H_{11}^2} - \sqrt{r^2 - (S - V)^2}\right)^2 + [(S - V) - H_{11}]^2} - \dfrac{H_{21} - H_{12}}{2}}{2(R + W)}\right)\Bigg]\Bigg\} \\
& + \sqrt{(S - V)^2 + \left(Z + r - \sqrt{r^2 - (S - V)^2}\right)^2} \\
& \cdot \sin^{-1}\Bigg\{\tan^{-1}\left(\frac{Z + r - \sqrt{r^2 - (S - V)^2}}{(S - V)}\right) + \tan^{-1}\left(\frac{\sqrt{r^2 - (S - V)^2} - \sqrt{r^2 - V^2}}{S}\right) \\
& - \Bigg[180^\circ - \Bigg[\Bigg[\frac{\left(180^\circ - 2\sin^{-1}\dfrac{\sqrt{S^2 + \left(\sqrt{r^2 - (S - V)^2} - \sqrt{r^2 - V^2}\right)^2}}{2(R + W)}\right)}{2}\Bigg] \\
& + \Bigg[90^\circ - \Bigg[\theta - \left(\sin^{-1}\left(\frac{H_{21}}{R + W}\right) + 2\sin^{-1}\right. \\
& \times \left.\left(\frac{\sqrt{\left(\sqrt{r^2 - H_{11}^2} - \sqrt{r^2 - (S - V)^2}\right)^2 + [(S - V) - H_{11}]^2} - \dfrac{H_{23} - H_{12}}{2}}{2(R + W)}\right)\right)\Bigg]\Bigg]\Bigg]\Bigg]\Bigg\}
\end{aligned}
$$

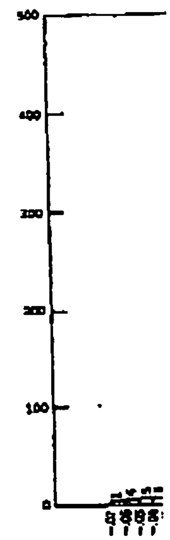
Figure 7-5

form. The various dimensions were assumed to be independent and equally likely between their upper and lower tolerance limits. The result of 8000 simulations is shown in Figure 7-6.

Note that the results appear nearly normal and that interference occurred 71 times in 8000 simulations. Based on these results it was decided to produce units without a design change and to rebuild those few on which interference did occur. The results of the actual assembly operation corresponded very well with the prediction that 71/8000 would not fit.

Summary

Tolerances of parameters can be combined by Monte Carlo methods or by Taylor series approximation. Both produce the probability of being outside certain limits and are less conservative than deterministic tolerance studies. Note that the analysis described above is the basis for deciding which tolerances to assign to what parts.



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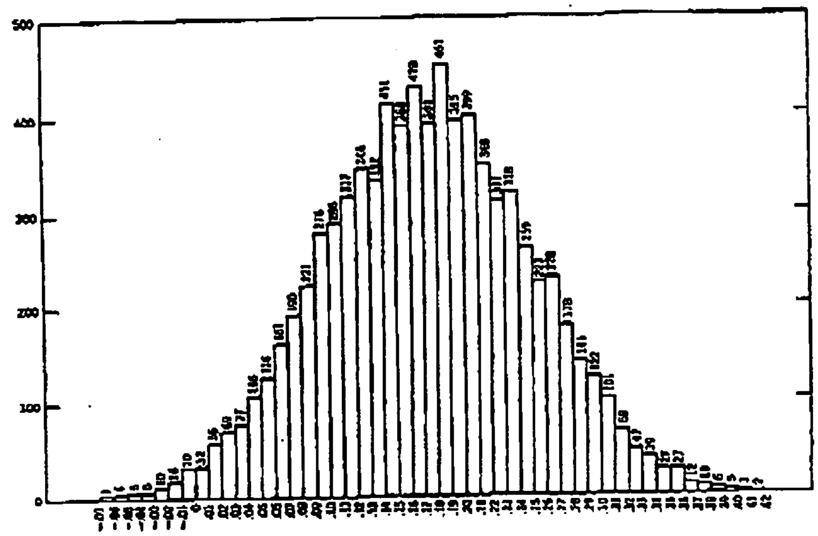

Figure 7-6

7-3 CLOSURE TIME OF SWITCHING CIRCUITS

Consider that the distribution of closure times of switches (e.g., relays, fluid logic, solid state devices) is known. We wish to know the closure time of networks of these switches.

We first model the circuit with a mathematical function and then solve the basic mathematical problems in more general terms (so we can use it later).

Consider two switches in series as shown in Figure 7-7. What is the closure time T of the series circuit in terms of T_1, closure time of the first switch and T_2, closure time of the second switch?

There will be a closed circuit from a to b at the time when both switches are closed or when the latter of the two switches closes. That is

$$T = \max(T_1, T_2).$$

Figure 7-7

median value of the conditional distribution of β given $\mathbf{Y}$, $f(\beta|\mathbf{Y})$. If the density $f(\beta|\mathbf{Y})$ in addition to being symmetric is also unimodal, the mean, median, and mode will all be at the *same* location. Hence when $f(\beta|\mathbf{Y})$ is symmetric about the parameter vector β and is also unimodal, b_{SEL} is b_{MAP}. When the distribution is *not* symmetric or *not* unimodal b_{SEL} and b_{MAP} are rarely the same. Some nonsymmetric unimodal probability densities are depicted in Fig. 4.2. Note that the modes do not coincide with the means. This causes the parameters b_{SEL} given by (4.7.3) and associated with the *mean* to be *not* equivalent to those given by the *mode* which are indicated by (4.7.2).

The conditional probability density $f(\beta|\mathbf{Y})$ used in (4.7.2) can be written in terms of other densities using the form of Bayes's theorem written as

$$f(\beta|\mathbf{Y})=\frac{f(\mathbf{Y}|\beta)f(\beta)}{f(\mathbf{Y})} \tag{4.7.4}$$

The probability density $f(\beta)$ contains the prior information known regarding the parameter vector β. Notice that the parameters appear only in the numerator of the right side of (4.7.4); this numerator can also be written as

$$f(\mathbf{Y},\beta)=f(\mathbf{Y}|\beta)f(\beta) \tag{4.7.5}$$

Then the necessary conditions given by (4.7.2) can be written equivalently as

$$\left.\frac{\partial \ln[f(\mathbf{Y},\beta)]}{\partial \beta_i}\right|_{b_{MAP}}=\left.\frac{\partial \ln[f(\mathbf{Y}|\beta)]}{\partial \beta_i}\right|_{b_{MAP}}+\left.\frac{\partial \ln[f(\beta)]}{\partial \beta_i}\right|_{b_{MAP}}=0 \tag{4.7.6}$$

since the maximum of $f(\mathbf{Y},\beta)$ exists at the same location as the maximum of its natural logarithm.

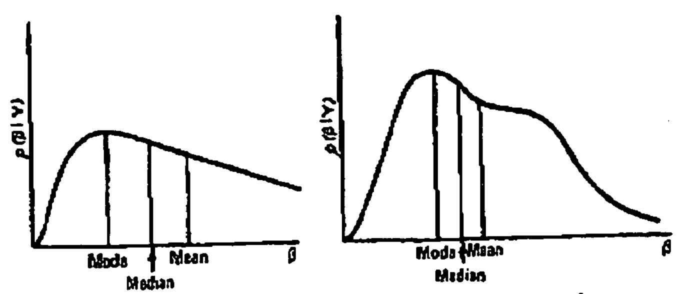

Figure 4.2 Some nonsymmetric conditional probability densities.

The estimators b_{MAP}, b_{SEL} are described without reference to the linearity or nonlinearity of the expected value of Y in the β's nor to the independence of the Y_i's. Under some assumptions about the structure of η_i and under some assumptions about the prior distribution of the β's, the MAP and SEL procedures are equivalent in arithmetic to certain least squares or Gauss–Markov procedures.

4.8 COST

Methods of collecting data and analyzing them must be coordinated. If observations are expensive, sophisticated methods of analysis to extract all pertinent information are justified. Sometimes more expensive methods of collecting data yield net returns by drastically reducing the cost of analysis. Increased costs due to collecting more data or using more sophisticated methods of analysis may or may not reduce the cost occasioned by the degree to which the estimate is incorrect. Some remarks in Chapter 3 were directed to these matters.

4.9 MONTE CARLO METHODS

One method for investigating the effects of nonlinearity or various other effects that are difficult to analyze otherwise is called the Monte Carlo method. Actually, what we describe is sometimes referred to as the "crude" Monte Carlo method. More sophisticated Monte Carlo methods often provide the same amount of information as the crude method but at a lower cost [1].

The Monte Carlo method can be used to investigate analytically the properties of a proposed estimation method. To simulate a series of experiments on the computer we proceed as follows:

1. Define the system by prescribing (*a*) the model equation, also called regression function, (*b*) the way in which "errors" are incorporated in the model of the observations, (*c*) the probability distribution of all the errors and, where applicable, (*d*) a prior distribution. Assign "true" values to all the parameters (β) in the regression function and to those in the distribution of error.
2. Select a set of values of the independent variables. Then calculate the associated set of "true" values of η from the regression equations.
3. Use the computer to produce a set of errors ϵ drawn from the prescribed probability distribution. For most computers programs are

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available which can generate a stream of numbers that have all the important characteristics of successive independent observations on a population uniform over the interval (0, 1). Since they are generated by a deterministic scheme, they are not actually random. Such numbers are called *pseudorandom numbers*. Suitable transformations are used to obtain samples for any other distribution.

To obtain a sequence of pseudorandom observations on a normal population with expected value 0 and variance 1, we can make use of the Box–Muller transformation [2]. If u_{2i-1} and u_{2i} are independent (0, 1) random numbers,

$$x_{2i-1} = (-2\ln u_{2i-1})^{1/2}\cos(2\pi u_{2i}) \tag{4.9.1a}$$

and

$$x_{2i} = (-2\ln u_{2i-1})^{1/2}\sin(2\pi u_{2i}) \tag{4.9.1b}$$

are independent random observations on a normal distribution with expected value 0 and variance 1. The normal random numbers are then adjusted to have the desired variances and covariances.

The simulated measurements are obtained by combining the errors with the regression values. For additive errors, the ith error is simply added to the ith η value. This then provides simulated measurements.

4. Acting as though the parameters are unknown, we estimate the parameters, denoting the estimates β^*.
5. Replicate the series of simulated experiments N times by repeating steps 3 and 4, each time with a new set of errors.
6. We use appropriate methods to estimate properties of the distribution of parameter estimates. (We consider the estimates actually obtained by our pseudorandom number scheme to be a random sample from the distribution of all possible estimates.) The expected value of our parameter estimator is estimated by the mean of our parameter estimates,

$$\bar{\beta}_j^* = \frac{1}{N}\sum_{i=1}^{N} \beta_{ji}^* \tag{4.9.2}$$

where β_{ji}^* is the jth component of the β^* found on the ith replication. If β^* may be a biased estimator, $\beta^* - \beta$ is an estimate of the bias. If it is not clear whether or not β^* is biased the size of $\beta^* - \beta$ needs to be compared with an estimate of its variance-covariance matrix.

The variances and covariances of the distribution of β^* may be

estimated by

$$\text{est. cov}(\beta_j^*, \beta_k^*) = \frac{1}{N-1}\sum_{i=1}^{N}\left(\beta_{ji}^* - \bar{\beta}_j^*\right)\left(\beta_{ki}^* - \bar{\beta}_k^*\right) \tag{4.9.3a}$$

If β^* is known to be unbiased, we can make use of our knowledge of β and use a slightly more efficient estimator

$$\text{est. cov}(\beta_j^*, \beta_k^*) = \frac{1}{N}\sum_{i=1}^{N}(\beta_{ji}^* - \beta_j)(\beta_{ki}^* - \beta_k) \tag{4.9.3b}$$

If β^* is biased, the right side of (4.9.3b) which are estimates of mean square error and corresponding product moments, may be more interesting than variances and covariances. If we use actual experiments rather than simulated ones (4.9.3b) will be not available although (4.9.2) and (4.9.3a) are.

The flexibility of the above simulation procedure is great. We can estimate the sample properties for any model, linear or nonlinear, and for any parameter values. We can estimate the effect of different probability distributions upon ordinary least squares estimation or other estimation methods. Many other possibilities also exist. An example of a Monte Carlo simulation is given below and another one is given in Section 6.9. These simulations can be accomplished on a modern high-speed computer at a small fraction of the cost, in time and money, of a comparable set of physical experiments.

The great power of the Monte Carlo procedure is that we can investigate the properties of estimators in cases for which the character of the estimators cannot be derived. To demonstrate the validity of a Monte Carlo procedure an example is considered which is simple enough to be analyzed without recourse to simulation. We investigate estimating β in the model $\eta_i = \beta X_i$ for the case of additive, zero mean, constant variance, uncorrelated errors; that is

$$y_i = \eta_i + \varepsilon_i, \qquad E(\varepsilon_i) = 0, \qquad V(\varepsilon_i) = \sigma^2, \qquad E(\varepsilon_i\varepsilon_j) = 0 \quad \text{for } i \neq j$$

The distribution of ε_i is uniform in the interval $(-.5, .5)$; each ε_i is found using a pseudorandom number generator. There are no errors in X_i and there is no prior information.

The X_i values are $X_i = i$ for $i = 1, 2, \ldots, 10$ and $\beta = 1$. For the kth set of simulated measurements, β_k^* is found using the ordinary least squares

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estimator,

$$\beta_k^* = \left[\sum_{i=1}^{10} X_i Y_{i,k}\right]\left[\sum_{i=1}^{10} X_i^2\right]^{-1}$$

The estimated expected value of β_k^*, (4.9.2), the estimated variance of β_k^*, (4.9.3a), and the estimated mean square error of β_k^*, (4.9.3b), are obtained by using

$$\bar{\beta}^* = \frac{1}{10}\sum_{k=1}^{10} \beta_k^*, \text{ est. } V(\beta^*) = \frac{1}{9}\sum_{k=1}^{10}\left(\beta_k^* - \bar{\beta}^*\right)^2$$

$$\text{est. mean square error } (\beta^*) = \frac{1}{10}\sum_{k=1}^{10}(\beta_k^* - 1)^2$$

For independent sets of errors, estimates were calculated for $N = 5, 25, 50, 100, 200$, and 500. The results are shown in Table 4.1 where the estimated standard deviation and estimated root mean square error are given rather than their squares. In Table 4.2 comparable results for a simulation involving normal errors are given. The variance of ϵ_i in this case was taken as $1/12$, the same as the variance for the uniform case.

In both Tables 4.1 and 4.2 the sample mean $\bar{\beta}^*$ tends to approach the ture value of 1 as N becomes large. Hence β^* is an unbiased estimator of β. Also the estimated standard error of β^* and estimated root mean square error tend to their common exact value

$$\left\{\sigma^2\left[\sum X_i^2\right]^{-1}\right\}^{1/2} = \left\{\frac{1/12}{385}\right\}^{1/2} = 0.014712$$

Table 4.1 Monte Carlo Simulation for $\eta_i = \beta X_i$, with $\beta = 1$ and $X_i = i$, $i = 1, 2, \ldots, 10$. Uniform Distribution of Errors

Sample Size	$\bar{\beta}^*$	Est. Std Dev (β^*)	Est. Root Mean Square Error (β^*)
5	1.0044	0.00950	0.00958
25	1.0014	0.01616	0.01589
50	0.9992	0.01350	0.01339
100	0.9996	0.01425	0.01418
200	1.0018	0.01440	0.01448
500	0.9987	0.01415	0.01419

Table 4.2 Monte Carlo Simulation for $\eta_i = X_i$, with $\beta = 1$ and $X_i = i$, $i = 1, 2, \ldots, 10$. Normal Distribution of Errors

Sample Size	$\bar{\beta}^*$	Est. Std Dev (β^*)	Est. Root Mean Square Error (β^*)
5	1.0021	0.01156	0.01055
25	0.9969	0.01608	0.01606
50	0.9972	0.01496	0.01507
100	0.9973	0.01486	0.01502
200	0.9995	0.01410	0.01407
500	0.9997	0.01480	0.01478

This example shows that the number of simulations N must be quite larg in order to provide accurate estimates of the variance of the paramete estimate. Such simulations are still inexpensive compared to actual experi ments to determine the variance. Moreover, methods are available fo making the simulation procedure more efficient [1].

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1. Hammersley, J. M. and Handscomb, D. C., *Monte Carlo Methods*, Methuen & Co. Ltd London, 1964.
2. Box, G. E. P. and Muller, M. E., "A Note on the Generation of Random Norma Deviates," *Ann. Math. Stat.*, 29 (1958), 610-611.